

Contradictory implications of the nonadditive entropy

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Abstract

It is shown that the concept of nonadditive black hole entropy leads to the contradictory implications in the framework of statistical thermodynamics. In particular, a black hole with the nonadditive entropy cannot be in thermal equilibrium with ordinary matter. Moreover, such black holes are mutually exclusive, i.e. they cannot compose a single system.

According to statistical mechanics the entropy of a thermodynamical system is the logarithm of the number of microstates accessible to it Γ , that is,

$$S = \ln \Gamma. \quad (1)$$

The black hole is a thermodynamical system with the Bekenstein-Hawking entropy

$$S_{\text{BH}} = \frac{A}{4l_P^2}, \quad (2)$$

where A is the area of the event horizon. A central problem in black hole physics is to express the Bekenstein-Hawking entropy in terms of the microstates (1). The essential reason for taking the logarithm in (1) is to make the entropy of a conventional system an *additive* quantity, for the statistical independent systems. In the literature there are however some doubts about additivity of black holes [1]. The point is that the Bekenstein-Hawking entropy is not a homogeneous first order function of the black hole energy.

Moreover, we cannot divide a black hole into two independent subsystems by a partition as an ideal gas in a box (the area theorem). And the black hole constituents cannot be extracted from a black hole. Therefore the black hole cannot be thought as made up of any constituent subsystems each of them endowed with its own independent thermodynamics; we have to consider a single black hole as a whole system.

In accord with this ideas I suggested in [2] that the statistical entropy of a black hole is not the logarithm of the number of microstates (1) but is proportional to this number

$$S_{bh} = 2\pi\Gamma, \quad (3)$$

where

$$\Gamma = \frac{A}{8\pi l_P^2}. \quad (4)$$

This means that the black hole is a *nonadditive* system.

In this note I argue that a black hole with the nonadditive entropy (3) cannot be in thermal equilibrium with ordinary matter. Moreover such black holes are mutually exclusive, i.e. they cannot compose a single system. I show that the concept of the nonadditive entropy (3) leads to the contradictory conclusions in the framework of the standard thermodynamics. This conclusion can be relevant for nonextensive statistical mechanics [3].

The argument is simple and goes as follows. Consider a black hole and ordinary matter in thermal equilibrium with each other, forming an isolated system [4]. For simplicity I consider a Schwarzschild black hole. Denote the energy and entropy of the black hole as E_{bh} and S_{bh} , and the energy and entropy of ordinary matter as E_{mat} and S_{mat} . Assume that 1) the principle of maximum entropy is valid and 2) the energies and entropies are additive for weakly coupled subsystems: $E = E_{bh} + E_{mat}$ and $S = S_{bh} + S_{mat}$. Then the entropy S of the system has its maximum value for a given energy E of the system. Since the total energy is fixed, S is really a function of one independent variable, say E_{bh} , and the necessary condition for a maximum may be written

$$\frac{dS}{dE_{bh}} = \frac{dS_{bh}}{dE_{bh}} + \frac{dS_{mat}}{dE_{mat}} \frac{dE_{mat}}{dE_{bh}} = 0 \quad (5)$$

or

$$\frac{dS_{bh}}{dE_{bh}} = \frac{dS_{mat}}{dE_{mat}}. \quad (6)$$

I also assume that 3) the entropy (3) satisfies the same thermodynamical relations as the Bekenstein-Hawking entropy, in particular $dS_{bh}/dE_{bh} = 1/T_H$, T_H being the Hawking temperature, $T_H = 1/(8\pi E_{bh})$. Then the condition for equilibrium is

$$T_H = T_{mat}. \quad (7)$$

This result is obtained from purely thermodynamical reasoning, without any statistical assumptions about the form of the entropy. Obviously it has the same form as for the conventional subsystems. Consider now the same condition from a statistical mechanical point of view. Denote the number of microstates accessible to the black hole and ordinary matter as Γ_{bh} and Γ_{mat} . Assume now that 1) the black hole and ordinary matter are statistically independent so that the number of microstates accessible to the whole system is

$$\Gamma = \Gamma_{bh}\Gamma_{mat}, \quad (8)$$

and 2) thermal equilibrium is realized by the greatest number of microstates, so that we can maximize this expression with respect to E_{bh} by writing

$$\frac{d\Gamma}{dE_{bh}} = \Gamma_{mat} \frac{d\Gamma_{bh}}{dE_{bh}} + \Gamma_{bh} \frac{d\Gamma_{mat}}{dE_{mat}} \frac{dE_{mat}}{dE_{bh}} = 0 \quad (9)$$

or

$$\frac{1}{\Gamma_{bh}} \frac{d\Gamma_{bh}}{dE_{bh}} = - \frac{1}{\Gamma_{mat}} \frac{d\Gamma_{mat}}{dE_{mat}}. \quad (10)$$

For the entropy of ordinary matter we have the standard formula with the logarithm (1). But for the black hole we have (3) and

$$\frac{d\Gamma_{bh}}{dE_{bh}} = \frac{1}{2\pi} \frac{dS_{bh}}{dE_{bh}} = \frac{1}{2\pi} \frac{1}{T_H}. \quad (11)$$

So

$$\frac{1}{2\pi\Gamma_{bh}} \frac{1}{T_H} = - \frac{1}{\Gamma_{mat}} \quad (12)$$

or

$$\frac{1}{S_{bh}} \frac{1}{T_H} = - \frac{1}{T_{mat}} \quad (13)$$

But $E_{bh} = 2T_H S_{bh}$. Then

$$\frac{E_{bh}}{2} = T_{mat} \quad (14)$$

or

$$\frac{1}{16\pi T_H} = T_{mat} \quad (15)$$

But this violates the zeroth law of thermodynamics for systems in thermal equilibrium. It is obvious that we cannot redefine the black hole temperature by simply setting $T_{bh} \equiv 1/(16\pi T_H)$. Moreover, this relation does not agree with (7). Thus the concept of thermal equilibrium cannot be formulated for black holes with the nonadditive entropy (3).

Note that this conclusion is valid not only for the formula (3) but also for the standard formula (1) if Γ equals (4). Moreover, the formula $S_{bh} = \ln(A/8\pi l_P^2)$ contradicts the second law of black hole thermodynamics [5], [2].

Consider now a system of black holes. Suppose, that two black holes are far apart and their interaction is negligible, so that they can be viewed as statistically independent. Let $S_{1(2)} = 2\pi\Gamma_{1(2)}$ and $\Gamma_{1(2)}$ be the entropy and degeneracy of the first (second) black hole, respectively. Then the number of states for the combined system is

$$\Gamma = \Gamma_1\Gamma_2. \quad (16)$$

What is the entropy of the system? Obviously, we cannot write the total entropy as $S = 2\pi(\Gamma_1\Gamma_2)$ because our system is not a single black hole. It seems that we would take the logarithm of Γ : $\ln \Gamma = \ln \Gamma_1 + \ln \Gamma_2$. But in this case, as mentioned above, we cannot interpret $\ln \Gamma_{1(2)}$ as the entropy of the first (second) black hole. Despite this failure the laws of thermodynamics are still valid, so we may define the total entropy as

$$S = S_1 + S_2 = 2\pi\Gamma_1 + 2\pi\Gamma_2 = 2\pi(\Gamma_1 + \Gamma_2), \quad (17)$$

whence

$$\Gamma_{total} = \Gamma_1 + \Gamma_2. \quad (18)$$

This means that these two black holes are mutually exclusive, i.e. no two black holes can be simultaneously in a single system. But this does not agree with (16). We can extend this conclusion to an arbitrary number of black holes. Note that additivity of entropies (17) is valid even when the subsystems cannot be considered independent and interact strongly among themselves; it is a consequence of the additivity of actions in a path integral approach to statistical thermodynamics [6].

Thus the concept of the nonadditive entropy leads to the contradictory conclusions in the framework of the standard thermodynamics.

In conclusion, the following point may be noted. In deriving (3) in [2], I used the concept of the internal (Euclidean) angular momentum of a black hole $L_z = A/8\pi$. Although its identification with the number of microstates (4) is not correct, this concept is well established. In [7], by following the approach used by Susskind [8] to derive the Rindler energy, I obtained quantization of the black hole area from the commutation relation and quantization condition for L_z . But L_z can be defined in more simple way from the Bunster-Carlip equation [9]

$$\frac{\hbar}{i} \frac{\partial \psi}{\partial \Theta} - \frac{A}{8\pi} \psi = 0, \quad (19)$$

where Θ is the lapse of the hyperbolic angle at the horizon. Analytically continuing Θ and $A/8\pi$ to the real values of $\Theta_E = i\Theta$ and $(A/8\pi)_E = -i(A/8\pi)$ we obtain

$$-i\hbar \frac{\partial \psi}{\partial \Theta_E} - \left(\frac{A}{8\pi} \right)_E \psi = 0, \quad (20)$$

As a result, $(A/8\pi)_E$ and Θ_E become conjugate. This means that the area is the operator-valued quantity, the angular momentum. Indeed, in the semi-classical approximation

$$\psi = a \exp \left(\frac{i}{\hbar} I \right), \quad (21)$$

where I is the action of a black hole. Substituting this in (19) we obtain

$$\frac{\partial I}{\partial \Theta} \psi = \frac{A}{8\pi} \psi; \quad (22)$$

the slowly varying amplitude a need not be differentiated. Under Euclidean continuation $\Theta_E = i\Theta$ and $(A/8\pi)_E = -i(A/8\pi)$,

$$\frac{\partial I}{\partial \Theta_E \psi} = \left(\frac{A}{8\pi} \right)_E \psi. \quad (23)$$

The derivative $\partial I / \partial \Theta_E$ is just a generalized momentum corresponding to the angle of rotation about one of the axes (say, the z^{th}) for a mechanical system. Therefore the operator $(A/8\pi)_E$ is what corresponds in quantum mechanics to the z component of angular momentum \hat{L}_z . Medved [10] found it immediately from the Bunster-Carlip action [9].

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